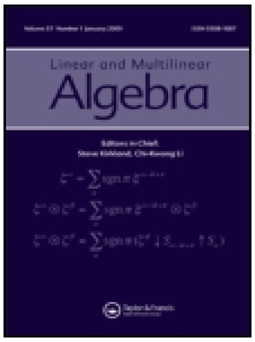

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Statistical Proofs of Some Matrix Results

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Using statistical arguments, proofs of some results in matrix theory are provided. Most of the proofs use results in linear statistical models and multivariate analysis. Some new results are also obtained in the process. In particular, several inequalities involving parallel sum of matrices are obtained.

1. INTRODUCTION AND NOTATIONS

Matrix theory has been extensively used in statistics, especially in multivariate analysis, linear models and design of experiments. The use of statistical arguments in proving results in matrix theory, however, has not received as much attention. Mitra (1973) proved some results on nonnegative definite matrices using statistical arguments. Mitra and Puntanen (1991) gave an interesting statistical interpretation of shorted operators. In a recent paper, Mitra (1993) used statistical arguments to derive explicitly the orthogonal projector onto the range of WP_1 where W is a given matrix and P_1 is a given orthogonal projector. In this communication we present a set of results in matrix theory whose proofs are based on statistical arguments.

To begin with, we introduce some notations. Throughout, all vectors and matrices are real, vectors being written as column vectors and denoted by bold face letters.

For a matrix A, $\mathcal{M}(A)$, $\det(A)$, A' will respectively denote the column span (range), determinant (if A is square) and transpose. Also, for a matrix A, A^- will denote a generalized inverse (g-inverse), i.e., A^- is a solution of the matrix equation $AXA = A$.

For a random vector $\mathbf{U}$, $E(\mathbf{U})$ and $D(\mathbf{U})$ denote respectively the expectation and dispersion (variance-covariance) matrix. For a scalar random variable, $\text{Var}(\cdot)$ denotes its variance and $\text{Cov}(\cdot,\cdot)$ stands for the covariance (matrix of covariances) of two random variables (vectors).

2. RESULTS

We shall often make use of the following well known result.

LEMMA 2.1 *If A is a symmetric, nonnegative definite (n.n.d.) matrix, then A is the dispersion matrix of some random vector. Conversely, the dispersion matrix of any random vector is n.n.d..*

Our first result is a Cauchy-Schwarz type inequality.

THEOREM 2.1 *Let* **a**, **b** *be non-null vectors and A, a symmetric n.n.d. matrix. Then,*

$$(\mathbf{a}'\mathbf{b})^2 \leq (\mathbf{a}'A^-\mathbf{a})(\mathbf{b}'A\mathbf{b}) \quad \forall \mathbf{a} \in \mathscr{M}(A). \tag{1}$$

Proof Since A is symmetric, n.n.d., we can write $A = X'X$, where X is a real matrix of order $m \times n$ (say). Consider the linear model

$$E(Y) = X\boldsymbol{\beta}, \quad D(Y) = I_m, \tag{2}$$

where I_m is the mth order identity matrix.

Since $\mathbf{a} \in \mathscr{M}(A)$, $\mathbf{a}'\boldsymbol{\beta}$ is estimable under the above model. The best linear unbiased estimator (BLUE) of $\mathbf{a}'\boldsymbol{\beta}$ is $\mathbf{a}'\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is a solution of the normal equation $A\boldsymbol{\beta} = X'Y$. Also

$$\mathrm{Var}(\mathbf{a}'\hat{\boldsymbol{\beta}}) = \mathbf{a}'A^-\mathbf{a}. \tag{3}$$

Observe that since $\mathbf{a} \in \mathscr{M}(A)$, $\mathbf{a}'A^-\mathbf{a}$ is invariant w.r.t. the choice of a g-inverse of A.

Again, since each component of $X\boldsymbol{\beta}$ is estimable under (2), $\mathbf{b}'X'X\boldsymbol{\beta}$ is also estimable under (2) for all $\mathbf{b} \neq 0$. Let $\mathbf{m} = X'X\mathbf{b} = A\mathbf{b}$. The variance of the BLUE of $\mathbf{m}'\boldsymbol{\beta}$ is

$$\begin{aligned} \mathrm{Var}(\mathbf{m}'\hat{\boldsymbol{\beta}}) &= \mathbf{m}'A^-\mathbf{m} \\ &= \mathbf{b}'AA^-A\mathbf{b} \\ &= \mathbf{b}'A\mathbf{b}. \end{aligned} \tag{4}$$

Finally,

$$\mathrm{Cov}(\mathbf{a}'\hat{\boldsymbol{\beta}}, \mathbf{m}'\hat{\boldsymbol{\beta}}) = \mathbf{a}'A^-\mathbf{m} = \mathbf{a}'\mathbf{b}, \tag{5}$$

where, in proving (5), we have used the fact that $\mathbf{a} \in \mathscr{M}(A)$. The result then follows from the well known correlation (Cauchy-Schwarz) inequality. Equality in (1) is attained if and only if $\mathbf{b}'A = c\mathbf{a}'$ for some constant c. ∎

The result of Theorem 2.1 for the special case $\mathbf{a} = \mathbf{b}$ was proved by Dey and Gupta (1977).

A characterization of n.n.d. matrices was given by Albert (1969), making use of purely matrix theoretic methods. We give a statistical proof of Albert's result.

THEOREM 2.2 *Let a symmetric matrix A be partitioned as*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{6}$$

where A_{11} and A_{22} are symmetric and $A'_{12} = A_{21}$. Then, A is n.n.d. if and only if

(*a*) A_{11} *is n.n.d.*

(*b*) $A_{22} - A_{21}A_{11}^{-}A_{12}$ *is n.n.d.*

(*c*) $\mathcal{M}(A_{12}) \subset \mathcal{M}(A_{11})$.

Proof $\Rightarrow$. Let $\mathbf{U}' = (\mathbf{U}'_1, \mathbf{U}'_2)$ be a normally distributed random vector with mean $\mathbf{0}$ and dispersion matrix A. Henceforth, we shall write this fact as $\mathbf{U} \sim N_n(\mathbf{0}, A)$. Here $\mathbf{U}_1$ is $p \times 1$, $\mathbf{U}_2$ is $q \times 1$, $p + q = n$, the order of A.

Partition A as in (6), where A_{11} is $p \times p$. Then since $D(\mathbf{U}_1) = A_{11}$, A_{11} is n.n.d., by Lemma 2.1. Also, the dispersion matrix of the conditional distribution of $\mathbf{U}_2$ given $\mathbf{U}_1$ is

$$D(\mathbf{U}_2|\mathbf{U}_1) = A_{22} - A_{21}A_{11}^{-}A_{12} = A_{22.1} \quad \text{(say)} \tag{7}$$

so that, by Lemma 2.1, $A_{22.1}$ is n.n.d..

Now, let $\mathbf{a}$ be a non-null vector such that $\mathbf{a}'A_{11}\mathbf{a} = 0$, which is equivalent to $\mathbf{a}'A_{11} = 0$, since A_{11} is n.n.d.. But, $\text{Var}(\mathbf{a}'U_1) = \mathbf{a}'A_{11}\mathbf{a}$. Thus

$$\begin{aligned} \mathbf{a}'A_{11} = 0 &\Rightarrow \mathbf{a}'U_1 = 0 \quad \text{almost surely (a.s.)} \\ &\Rightarrow \text{Cov}(\mathbf{a}'U_1, \mathbf{b}'U_2) = 0 \quad \forall \mathbf{b} \\ &\Rightarrow \mathbf{a}'A_{12}\mathbf{b} = 0 \quad \forall \mathbf{b} \\ &\Rightarrow \mathbf{a}'A_{12} = 0 \\ &\Rightarrow \mathcal{M}(A_{12}) \subset \mathcal{M}(A_{11}). \end{aligned}$$

$\Leftarrow$ Let $\mathbf{Z}_1 \sim N_p(0, A_{11})$ and $\mathbf{Z}_2 \sim N_q(0, A_{22.1})$. Define $\mathbf{U}_1 = \mathbf{Z}_1$, $\mathbf{U}_2 = A'_{12}A_{11}^{-}\mathbf{Z}_1 + \mathbf{Z}_2$. Then, $D(\mathbf{U}_1) = A_{11}$, $D(\mathbf{U}_2) = A'_{12}A_{11}^{-}A_{11}(A_{11}^{-})'A_{12} + A_{22.1}$. Also, since $\mathcal{M}(A_{12}) \subset \mathcal{M}(A_{11})$, we have $A_{12} = A_{11}B$ for some matrix B and therefore

$$\begin{aligned} D(\mathbf{U}_2) &= A'_{12}A_{11}^{-}A_{11}(A_{11}^{-})'A_{12} + A_{22.1} \\ &= B'A_{11}A_{11}^{-}A_{11}(A_{11})'A_{11}B + A_{22.1} \\ &= B'A_{11}B + A_{22.1}. \end{aligned}$$

But, $B'A_{11}B = B'A_{11}A_{11}^{-}A_{11}B = A_{21}A_{11}^{-}A_{12}$ and hence $D(\mathbf{U}_2) = A_{22}$. Finally,

$$\begin{aligned} \text{Cov}(\mathbf{U}_1, \mathbf{U}_2) &= A_{11}(A_{11}^{-})'A_{12} \\ &= A_{11}(A_{11}^{-})'A_{11}B = A_{11}B = A_{12}. \end{aligned}$$

Hence

$$D(\mathbf{U}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and by Lemma 2.1, the matrix on the rhs in n.n.d. ∎

The next result may be regarded as a determinantal version of the Cauchy-Schwarz inequality, which was obtained by Gaffke and Krafft (1977). We prove this result using a simpler technique.

THEOREM 2.3 *Let A, B be matrices of the same order. Then*

$$\{\det(A'B)\}^2 \le \det(A'A)\det(B'B). \tag{8}$$

Proof To avoid trivialities, we assume that $A'A$, $B'B$ and $A'B$ are nonsingular. Let $\mathbf{Y}$ be a random vector such that $D(\mathbf{Y}) = I$. Define $\mathbf{U}_1 = A\mathbf{Y}$ and $\mathbf{U}_2 = B\mathbf{Y}$. Then the dispersion matrix of $\mathbf{U}' = (\mathbf{U}'_1, \mathbf{U}'_2)$ is

$$\begin{pmatrix} A'A & A'B \\ B'A & B'B \end{pmatrix}.$$

From Lemma 2.1, the above matrix is n.n.d. From Theorem 2.2 this implies that

$$B'B - B'A(A'A)^{-1}A'B$$

is also n.n.d. Hence $\det(B'B) \ge \det(B'A)[\det(A'A)]^{-1}\det(A'B)$ and the result follows. Equality in (8) is achieved if and only if $B = AC$ for some nonsingular matrix C. ∎

The following result was proved by Mitra (1973). We present a proof, which is believed to be simpler than that of Mitra (1973).

THEOREM 2.4 *If A and B are n.n.d. matrices partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

If $\rho(A+B) = \rho(A_{11} + B_{11})$ then $\rho(A) = \rho(A_{11})$ and $\rho(B) = \rho(B_{11})$. Here $\rho(\cdot)$ stands for the rank of the matrix.

Proof Let $\mathbf{U} \sim N(0, A)$ and independent of $\mathbf{V} \sim N(0, B)$. Define $\mathbf{Z} = \mathbf{U} + \mathbf{V}$. Clearly $\mathbf{Z} \sim N(0, A+B)$. Since $\rho(A+B) = \rho(A_{11} + B_{11})$ it follows that there exists C such that $\mathbf{Z}_2 = C\mathbf{Z}_1$ i.e. $\mathbf{U}_2 + \mathbf{V}_2 = C\mathbf{U}_1 + C\mathbf{V}_1$ a.s.. Since $\mathbf{U}$ and $\mathbf{V}$ are independent it follows that $\mathbf{U}_2 = C\mathbf{U}_1$ and $\mathbf{V}_2 = C\mathbf{V}_1$ a.s.. Hence the result. ∎

Karlin and Rinotte (1977) proved the following result. We state their result in the form we need.

LEMMA 2.2 *Let $g_1, g_2 : R^n \to R$ such that*

$$[g_1(t_1) - g_1(t_2)][g_2(t_1) - g_2(t_2)] \ge 0$$

for all $t_1, t_2 \in R^n$. Also let $\mathbf{Z} \sim N(0, \Sigma)$, $\mathbf{U}' = (z_1^2, z_2^2, \ldots, z_n^2)$, where $\mathbf{Z}' = (z_1, z_2, \ldots, z_n)$. Then

$$E[g_1(\mathbf{U})g_2(\mathbf{U})] \ge E[g_1(\mathbf{U})]E[g_2(\mathbf{U})] \tag{9}$$

provided $(-\Sigma^{-1})$ has nonnegative off diagonals.

Based on the above lemma, we prove the following result.

THEOREM 2.5 *Let A_i for $i = 1, \ldots, n$ be n.n.d. matrices each of order k. Let there exist a matrix M such that $M'A_iM$ is diagonal for each $i = 1, 2, \ldots, n$. Also let g_i be all non* *decreasing functions on R. Then*

$$\int_{R^k} \frac{1}{(2\pi)^{n/2}} e^{-\mathbf{x}'\mathbf{x}/2} \prod_{i=1}^{n} g_i(\mathbf{x}'A_i\mathbf{x})d\mathbf{x} \geq \prod_{i=1}^{n} \int_{R^k} \frac{1}{(2\pi)^{n/2}} e^{-\mathbf{x}'\mathbf{x}/2} g_i(\mathbf{x}'A_i\mathbf{x})d\mathbf{x} \tag{10}$$

provided $-M'M$ *has nonnegative off diagonals.*

Proof Let $M'A_iM = \Delta_i$. Then by making linear transformations, (10) can be written as

$$\int_{R^k} \frac{1}{(2\pi)^{1/2}} e^{-\mathbf{x}'\Sigma^{-1}\mathbf{x}/2} \prod_{i=1}^{n} g_i(\mathbf{x}'\Delta_i\mathbf{x})d\mathbf{x} \geq \prod_{i=1}^{n} \int_{R^k} \frac{1}{(2\pi)^{1/2}} e^{-\mathbf{x}'\Sigma^{-1}\mathbf{x}/2} g_i(\mathbf{x}'\Delta_i\mathbf{x})d\mathbf{x}$$

with $\Sigma = (M'M)^{-1}$. Now the result follows from the lemma. ■

The following results are on some matrix inequalities.

THEOREM 2.6 *Let A and B be any n.n.d matrices. Then*

$$[\mathbf{a}'A^{-}\mathbf{a}][\mathbf{a}'B^{-}\mathbf{a}] \geq [\mathbf{a}'(A+B)^{-}\mathbf{a}][\mathbf{a}'A^{-}\mathbf{a} + \mathbf{a}'B^{-}\mathbf{a}]. \tag{11}$$

provided $\mathbf{a} \in \mathcal{M}(A) \cap \mathcal{M}(B)$. *Equality is achieved if either A or B is of rank one.*

Proof Let $X_1'X_1 = A$ and $X_2'X_2 = B$. Consider the three linear models, $E\mathbf{Y}_i = X_i\boldsymbol{\beta}$ for $i = 1, 2, 3$, where $\mathbf{Y}_3' = (\mathbf{Y}_1', \mathbf{Y}_2')$ and $X_3' = (X_1', X_2')$.

Since $\mathbf{a} \in \mathcal{M}(A) \cap \mathcal{M}(B)$, $\mathbf{a}'\boldsymbol{\beta}$ is estimable under all the three models. The BLUE's of $\mathbf{a}'\boldsymbol{\beta}$ under the three models are $T_i = \mathbf{a}'\hat{\boldsymbol{\beta}}_i$ for $i = 1, 2, 3$, respectively, where $\hat{\boldsymbol{\beta}}_i = (X_i'X_i)^{-}X_i'\mathbf{Y}_i$ for $i = 1, 2, 3$. Also note that

$$\mathbf{a}'(A+B)^{-}\mathbf{a} = \mathrm{Var}(T_3) \leq \frac{1}{(w_1 + w_2)^2} \mathrm{Var}(w_1T_1 + w_2T_2)$$

$$= \frac{1}{(w_1 + w_2)^2}[w_1^2 \mathrm{Var}(T_1) + w_2^2 \mathrm{Var}(T_2)],$$

where $w_i^{-1} = \mathrm{Var}(T_i)$ for $i = 1, 2$. But $\mathrm{Var}(T_1) = \mathbf{a}'A^{-}\mathbf{a}$ and $\mathrm{Var}(T_2) = \mathbf{a}'B^{-}\mathbf{a}$. Hence

$$\mathbf{a}'(A+B)^{-}\mathbf{a} \leq \frac{[\mathbf{a}'A^{-}\mathbf{a}][\mathbf{a}'B^{-}\mathbf{a}]}{[\mathbf{a}'A^{-}\mathbf{a}] + [\mathbf{a}'B^{-}\mathbf{a}]}. \tag{12}$$

If either A or B is of rank one, T_3 is same as $(w_1T_1 + w_2T_2)/(w_1 + w_2)$. This completes the proof. ■

Remark 2.1 In fact, by using the same argument we can generalize the above result as follows:
if $\mathbf{a} \in \mathcal{M}(A+B)$ then

$$\mathbf{a}'(A+B)^{-}\mathbf{a} = \min_{\mathbf{a}_1 \in \mathcal{M}(A); \mathbf{a}_2 \in \mathcal{M}(B); \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}} [\mathbf{a}_1'A^{-}\mathbf{a}_1 + \mathbf{a}_2'B^{-}\mathbf{a}_2], \tag{13}$$

and that if, $\mathbf{a}\in\mathcal{M}(A)\cap\mathcal{M}(B)$

$$\mathbf{a}'(A+B)^{-}\mathbf{a}=\min_{\mathbf{a}_1\in\mathcal{M}(A);\mathbf{a}_2\in\mathcal{M}(B);\mathbf{a}_1+\mathbf{a}_2=\mathbf{a}}[\mathbf{a}_1'A^{-}\mathbf{a}_1+\mathbf{a}_2'B^{-}\mathbf{a}_2], \tag{14}$$

$$\leq\frac{[\mathbf{a}'A^{-}\mathbf{a}][\mathbf{a}'B^{-}\mathbf{a}]}{[\mathbf{a}'A^{-}\mathbf{a}]+[\mathbf{a}'B^{-}\mathbf{a}]}. \tag{15}$$

Remark 2.2 It should be noted that the above theorem can obviously be generalised in the following way:
Let A_i be n.n.d. matrices for $i=1,2,\ldots,k$ and $\mathbf{a}\in\cap A_i$ then

$$\mathbf{a}'\left(\sum_{i=1}^{k}A_i\right)^{-}\mathbf{a}\leq\frac{\sum_{i=1}^{k}1/[\mathbf{a}'A_i^{-}\mathbf{a}]^2}{[\sum_{i=1}^{k}1/(\mathbf{a}'A_i^{-}\mathbf{a})]^2}. \tag{16}$$

The following result follows from Theorem 2.6.

COROLLARY 2.1 *Let A be a n.n.d. symmetric matrix. Then, for any choice of a g-inverse,*

$$\mathbf{a}'(A+\mathbf{a}\mathbf{a}')^{-}\mathbf{a}=\mathbf{a}'A^{-}\mathbf{a}/(1+\mathbf{a}'A^{-}\mathbf{a}),\quad \textit{if}\quad \mathbf{a}\in\mathcal{M}(A)$$
$$=1,\quad \textit{otherwise}.$$

We now give an inequality involving parallel sum of matrices. We have the following definition.

DEFINITION 2.1 *The parallel sum of positive definite matrices A_i, for $i=1,2,\ldots,k$ is*

$$\prod_i:A_i=\left(\sum_i A_i^{-1}\right)^{-1}.$$

We prove the well-known series-parallel inequality (see for e.g. Anderson and Duffin (1969) Anderson, Morley and Trapp (1984)) using statistical arguments.

THEOREM 2.7 *Let A_{ij}, for $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$ be positive definite matrices. Then*

$$\sum_{j=1}^{m}\prod_{i=1}^{n}:A_{ij}\leq\prod_{i=1}^{n}:\sum_{j=1}^{n}A_{ij}. \tag{17}$$

Proof Let $\mathbf{U}_{ij}\sim N(\boldsymbol{\mu}_i,A_{ij})$ be independent observations. Define

$$\mathbf{Y}_i=\left(\sum_{j=1}^{m}A_{ij}^{-1}\right)^{-1}\sum_{j=1}^{m}A_{ij}^{-1}\mathbf{U}_{ij}$$

for $i=1,\ldots,n$. Then $\mathbf{T}=(\mathbf{Y}_1,\mathbf{Y}_2,\ldots,\mathbf{Y}_n)$ is complete sufficient for $\boldsymbol{\mu}_i$'s, and hence BLUE's of $\boldsymbol{\mu}_i$'s (in fact MVUE of $\boldsymbol{\mu}_i$'s). Let $\boldsymbol{\mu}=\sum_{i=1}^{n}\boldsymbol{\mu}_i$. Then BLUE of $\boldsymbol{\mu}$ is given by $\hat{\boldsymbol{\mu}}=\sum_{i=1}^{n}\mathbf{Y}_i$. It is easy to see that

$$D(\hat{\boldsymbol{\mu}})=\sum_{i=1}^{n}\left(\sum_{j=1}^{m}A_{ij}^{-1}\right)^{-1}. \tag{18}$$

Now consider another estimator of $\boldsymbol{\mu}$. Let $\mathbf{Z}_j=\sum_{i=1}^{n}\mathbf{U}_{ij}$. Hence $\mathbf{Z}_j\sim N(\boldsymbol{\mu},B_j)$, $B_j=\sum_{i=1}^{n}A_{ij}$, are independent. Define $\mathbf{Z}=(\sum_{j=1}^{m}B_j^{-1})^{-1}\sum_{j=1}^{m}B_j^{-1}\mathbf{Z}_j$, which is an unbiased estimator of $\boldsymbol{\mu}$, based on $\mathbf{Z}_j$ alone. Hence

$$D(\mathbf{Z}) \geq D(\hat{\boldsymbol{\mu}}), \tag{19}$$

where for a pair of n.n.d. matrices A and B, $A \geq B$ means $A - B$ is n.n.d..
But

$$D(\mathbf{Z}) = \left(\sum_{j=1}^{m} B_j \right)^{-1}. \tag{20}$$

From the equations (18), (19) and (20) the result follows. ∎

A similar results can be proved for n.n.d. (possibly singular) matrices, as given below.

THEOREM 2.8 *If A_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ are n.n.d. matrices then*

$$\mathbf{a}' \sum_{i=1}^{n} \left[\sum_{j=1}^{m} A_{ij} \right]^{-} \mathbf{a} \leq \mathbf{a}' \left(\sum_{j=1}^{m} H_j \right)^{-1} \left[\sum_{j=1}^{m} H_j \left(\sum_{i=1}^{n} A_{ij} \right) H'_j \right] \left(\sum_{j=1}^{m} H'_j \right)^{-1} \mathbf{a}, \tag{21}$$

provided $\mathcal{M}(H_j) \subset \cap_{i=1}^{n} \mathcal{M}(A_{ij})$ for every j and $\sum_{j=1}^{m} H_j$ is invertible.

Proof Consider the collection of linear models

$$\mathbf{Y}_{ij} = X_{ij} \boldsymbol{\beta}_i + \varepsilon_{ij} \tag{22}$$

where $A_{ij} = X'_{ij} X_{ij}$, and $\mathbf{Y}_{ij}$'s are independent. Define $\gamma = \sum_{i=1}^{n} \boldsymbol{\beta}_i$. The BLUE of $\mathbf{a}'\gamma$ is $\sum_{i=1}^{n} \mathbf{a}'\hat{\boldsymbol{\beta}}_i$, where $\mathbf{a} \in \cap_{i=1}^{n} \mathcal{M}(A_{ij})$ for every j. Hence $\mathrm{Var}(\mathbf{a}'\hat{\gamma})$ is

$$\sum_{i=1}^{n} \mathrm{Var}(\mathbf{a}'\hat{\boldsymbol{\beta}}_i) = \sum_{i=1}^{n} \mathbf{a}' \left(\sum_{j=1}^{n} A_{ij} \right)^{-} \mathbf{a}.$$

Now, let us look at the another estimator of γ. For every j let us consider an estimator of γ say $\hat{\gamma}_{(j)}$ based of $\mathbf{Y}_{ij}$ for $i = 1, 2, \ldots, n$, Then, the pseudo-dispersion matrix of $\hat{\gamma}_{(j)}$ is

$$D(\hat{\gamma}_{(j)}) = \sum_{i=1}^{n} A_{ij}^{-}.$$

Define

$$\hat{\gamma}_{II} = \left(\sum_{j=1}^{m} H_j \right)^{-1} \sum_{j=1}^{m} H_j \hat{\gamma}_{(j)}$$

Clearly $\mathrm{Var}(\mathbf{a}'\hat{\gamma})$ is not larger than $\mathrm{Var}(\mathbf{a}'\hat{\gamma}_{II})$. Hence the result. ∎

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